

# Rational Relativistic Collisions

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## Abstract

If two point particles collide relativistically in one dimension, and the masses, velocities and gamma factors of the incoming particles are rational numbers, then the velocities and gamma factors of the outgoing particles are rational. Numerous examples can be found using Pythagorean triples. At all velocities, the collision results in a Lorentzian reflection of the 2-momenta of the particles.

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# 1 Introduction

One of the most interesting problems in elementary relativity is to find the outcome of a 1-dimensional elastic collision of two particles [1, 2]. The masses and incoming velocities of the particles are assumed known. The masses are unchanged in the collision, and the problem is to find the outgoing velocities. The physical principle governing the collision is relativistic 2-momentum conservation, combining the requirements of relativistic energy and momentum conservation. This gives two equations for the outgoing velocities. The solution is different from what one gets using non-relativistic, Newtonian energy and momentum conservation. Experiments show that for particles colliding at speeds comparable to the speed of light, the relativistic results are correct. We set the speed of light to be 1.

The conservation equations are rather intimidating, because of gamma factors in each term. For velocity  $u$ , the gamma factor is  $\gamma(u) = (1 - u^2)^{-\frac{1}{2}}$ . It is easy to get bogged down trying to eliminate the square roots. One might expect that having done this, the resulting algebraic equations would be quadratic or quartic, and their solution would involve at least one square root. However, the text-book solution shows that the energy of one of the outgoing particles is a simple rational function of the incoming energies and momenta. Here we go further, and show that if the mass and velocity data of the incoming particles are all rational, in a sense that we will clarify, then the outgoing data are rational too.

It is easy to see that in a non-relativistic elastic collision, rational incoming data implies rational outgoing data. Suppose the incoming particles have masses  $m_1$  and  $m_2$ , and velocities  $u_1$  and  $u_2$ , and suppose the outgoing velocities are  $v_1$  and  $v_2$ . Newtonian energy and momentum conservation require that

$$m_1 u_1^2 + m_2 u_2^2 = m_1 v_1^2 + m_2 v_2^2 \quad (1)$$

and

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2, \quad (2)$$

where we have eliminated the  $\frac{1}{2}$  in the kinetic energies. Superficially, it appears that in solving for  $v_1$  and  $v_2$ , we will encounter square roots.

It is convenient to work in an inertial frame where the second particle is initially at rest, so  $u_2 = 0$ . Then, to find  $v_2$  we take the terms involving  $v_2$  to the left hand side, and eliminate  $v_1$  by squaring the second equation and subtracting  $m_1$  times the first. The result is a quadratic equation

$$(m_1 + m_2)v_2^2 - 2m_1 u_1 v_2 = 0, \quad (3)$$

but there is no term independent of  $v_2$ . One solution is therefore  $v_2 = 0$ , and this is because one possible outcome, consistent with energy and momentum conservation, is that the particles miss and retain their initial velocities. The quadratic equation (3) therefore reduces to a linear equation, with solution

$$v_2 = \frac{2m_1}{m_1 + m_2} u_1. \quad (4)$$

The solution for general  $u_2$  is found by performing a Galilean boost to make  $u_2 = 0$ , then determining  $v_2$  using eq.(4), and finally reversing the boost.  $v_1$  is then easily found using eq.(2). The final, rational expressions for  $v_1$  and  $v_2$  are very similar, because of particle exchange symmetry. They are

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 + \frac{2m_2}{m_1 + m_2} u_2, \quad v_2 = \frac{2m_1}{m_1 + m_2} u_1 + \frac{m_2 - m_1}{m_1 + m_2} u_2. \quad (5)$$

The outgoing velocities are therefore rational if the masses and incoming velocities are all rational.

Before looking at the details of relativistic collisions, we recall certain aspects of the rational geometry of a unit circle and hyperbola, and the role of Pythagorean triples.

## 2 Rational points on a circle and hyperbola

It will be helpful to recall three parametrisations of a circle, and the related parametrisations of a hyperbola. Consider the unit circle

$$x^2 + y^2 = 1. \quad (6)$$

The first parametrisation, using just  $y$ , is  $(x, y) = ((1 - y^2)^{\frac{1}{2}}, y)$ .  $y$  is between  $-1$  and  $1$ , and both square roots are needed. The second is the trigonometrical parametrisation  $(x, y) = (\cos \theta, \sin \theta)$ , with  $\theta$  between  $-\pi$  and  $\pi$ . Finally, there is the rational parametrisation

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}, \quad (7)$$

with  $t$  between  $-\infty$  and  $\infty$ . The relation of  $t$  to  $\theta$  is  $t = \tan \frac{1}{2}\theta$ .

Rational points on the circle are those with both  $x$  and  $y$  rational, and therefore  $\cos \theta$  and  $\sin \theta$  both rational. They are the points for which  $t$  is

rational, so they are dense on the circle. If  $t = \frac{N}{M}$ , with  $M$  and  $N$  integers, then

$$x = \cos \theta = \frac{M^2 - N^2}{M^2 + N^2}, \quad y = \sin \theta = \frac{2MN}{M^2 + N^2}. \quad (8)$$

A rational point is therefore related to a Pythagorean triple, a triple of integers  $(A, B, C) = (M^2 - N^2, 2MN, M^2 + N^2)$ , for which  $A^2 + B^2 = C^2$ . Pythagorean triples that differ by a constant integer multiple, e.g.  $(3, 4, 5)$  and  $(6, 8, 10)$ , correspond to the same point on the circle, so rational points are uniquely related to primitive Pythagorean triples. Examples of rational points are

$$(x, y) = \left( \frac{3}{5}, \frac{4}{5} \right) \quad \text{and} \quad (x, y) = \left( \frac{5}{13}, \frac{12}{13} \right), \quad (9)$$

obtained from the triples  $(3, 4, 5)$  with  $(M, N) = (2, 1)$ , and  $(5, 12, 13)$  with  $(M, N) = (3, 2)$ .

The rational points form a group, isomorphic to the rational subgroup of  $\text{SO}(2)$ . This is the group of matrices

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (10)$$

with  $\cos \theta$  and  $\sin \theta$  both rational. Each rational point  $(x, y)$  (written as a column vector) is obtained by acting with a rational rotation matrix on the point  $(1, 0)$ . The group law is addition of angles, even though the angles are not rational.

A similar analysis can be applied to the hyperbola

$$x^2 - y^2 = 1. \quad (11)$$

We only consider the branch with  $x \geq 1$  (having in mind the positive energy branch of the relativistic mass-shell condition). Two parametrisations are  $(x, y) = ((1 + y^2)^{\frac{1}{2}}, y)$ , with  $y$  between  $-\infty$  and  $\infty$  and the square root positive, and the parametrisation  $(x, y) = (\cosh \theta, \sinh \theta)$ , using the hyperbolic functions, with  $\theta$  between  $-\infty$  and  $\infty$ . Finally, there is the rational parametrisation

$$x = \frac{1 + t^2}{1 - t^2}, \quad y = \frac{2t}{1 - t^2}, \quad (12)$$

with  $t$  between  $-1$  and  $1$ . The relation of  $t$  to  $\theta$  is  $t = \tanh \frac{1}{2}\theta$ . There is a second rational parametrisation

$$x = \frac{1 + s^2}{2s}, \quad y = \frac{1 - s^2}{2s}, \quad (13)$$

but this is rationally obtained from the first by setting  $t = \frac{1-s}{1+s}$ .

By setting  $t = \frac{N}{M}$ , with  $M^2 > N^2$ , one obtains the rational point

$$x = \cosh \theta = \frac{M^2 + N^2}{M^2 - N^2}, \quad y = \sinh \theta = \frac{2MN}{M^2 - N^2}. \quad (14)$$

This is again related to the Pythagorean triple  $(M^2 - N^2, 2MN, M^2 + N^2)$ . Examples of rational points are  $(x, y) = (\frac{5}{3}, \frac{4}{3})$  and  $(\frac{5}{4}, \frac{3}{4})$ , related to the triples with  $(M, N) = (2, 1)$  and  $(3, 1)$ .

The rational points are dense on the hyperbola. They form a group isomorphic to the rational subgroup of  $\text{SO}(1,1)$ , the group of matrices

$$\begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} \quad (15)$$

with  $\cosh \theta$  and  $\sinh \theta$  both rational. Each rational point is obtained by acting with a matrix of this type on the point  $(1, 0)$ .

The rational points can also be understood by writing the equation for the hyperbola as  $(x - y)(x + y) = 1$ . The rational solutions are

$$(x - y, x + y) = \left( \frac{M - N}{M + N}, \frac{M + N}{M - N} \right), \quad (16)$$

and adding and subtracting these gives the formulae (14).

### 3 Rational masses and 2-velocities

We next recall the relativistic energy and momentum of a particle with mass  $m$  and velocity  $u$  moving in one dimension. Associated with the velocity is a (positive) gamma factor  $\gamma(u) = (1 - u^2)^{-\frac{1}{2}}$ .

The 2-velocity  $U$  of the particle is

$$U = (\gamma(u), \gamma(u)u). \quad (17)$$

This is a 2-vector for which the Lorentzian inner product  $U \cdot U = (\gamma(u))^2 - (\gamma(u)u)^2$  is unity, which follows immediately from the definition of  $\gamma(u)$ . So  $U$  is on the hyperbola  $x^2 - y^2 = 1$  that we discussed in the last section, and on the branch with  $x$  positive. We can use the parametrisation in terms of hyperbolic functions and set  $\gamma(u) = \cosh \theta$  and  $\gamma(u)u = \sinh \theta$ ; then the 2-velocity is  $U = (\cosh \theta, \sinh \theta)$ , and  $u = \tanh \theta$ .  $\theta$  is called the rapidity of the particle.

We say a 2-velocity is rational if  $\gamma(u)$  and  $\gamma(u)u$  are rational, so  $u$  is too. Equivalently,  $\sinh \theta$ ,  $\cosh \theta$  and  $\tanh \theta$  are all rational, and this requires  $t = \tanh \frac{1}{2}\theta$  to be rational. The velocities  $u$  giving rational 2-velocities are dense between  $-1$  and  $1$ . We deduce from eqs.(14) that for a rational 2-velocity,  $u$  and  $\gamma(u)$  can be expressed in terms of the integers  $(M, N)$  of a Pythagorean triple, with  $M^2 > N^2$ , as

$$u = \frac{2MN}{M^2 + N^2}, \quad \gamma(u) = \frac{M^2 + N^2}{M^2 - N^2}, \quad (18)$$

so  $\gamma(u)u = \frac{2MN}{M^2 - N^2}$ . The examples  $(M, N) = (2, 1)$ ,  $(3, 1)$  and  $(3, 2)$  give, respectively,  $(u, \gamma(u)) = (\frac{4}{5}, \frac{5}{3})$ ,  $(\frac{3}{5}, \frac{5}{4})$  and  $(\frac{12}{13}, \frac{13}{5})$ .

The relativistic 2-momentum of a particle with mass  $m$  and 2-velocity  $U$  is  $mU$ . Its components are written as

$$mU = (m\gamma(u), m\gamma(u)u), \quad (19)$$

where the first component is the relativistic energy and the second is the relativistic, spatial momentum. These satisfy the mass-shell condition  $(m\gamma(u))^2 - (m\gamma(u)u)^2 = m^2$ . In terms of the particle mass  $m$  and rapidity  $\theta$ ,

$$mU = (m \cosh \theta, m \sinh \theta). \quad (20)$$

The 2-momentum is rational if both the mass and 2-velocity are rational.

## 4 Relativistic collisions

We can now investigate the main problem, the relativistic elastic collision of two particles with masses  $m_1$  and  $m_2$ , and given initial velocities  $u_1$  and  $u_2$ . We use the hyperbolic parametrisation,  $u_1 = \tanh \phi_1$  and  $u_2 = \tanh \phi_2$ . The initial 2-momenta of the particles are  $m_1 U_1 = (m_1 \cosh \phi_1, m_1 \sinh \phi_1)$  and  $m_2 U_2 = (m_2 \cosh \phi_2, m_2 \sinh \phi_2)$ . The final, unknown velocities are denoted  $v_1 = \tanh \chi_1$  and  $v_2 = \tanh \chi_2$ , and the corresponding final 2-momenta are  $m_1 V_1 = (m_1 \cosh \chi_1, m_1 \sinh \chi_1)$  and  $m_2 V_2 = (m_2 \cosh \chi_2, m_2 \sinh \chi_2)$ . The initial data are rational if  $m_1, m_2, \cosh \phi_1, \sinh \phi_1, \cosh \phi_2$  and  $\sinh \phi_2$  are all rational.

The physical principle determining  $v_1$  and  $v_2$  is the conservation of total 2-momentum,

$$m_1 U_1 + m_2 U_2 = m_1 V_1 + m_2 V_2. \quad (21)$$

In components,

$$\begin{aligned} m_1 \cosh \phi_1 + m_2 \cosh \phi_2 &= m_1 \cosh \chi_1 + m_2 \cosh \chi_2, \\ m_1 \sinh \phi_1 + m_2 \sinh \phi_2 &= m_1 \sinh \chi_1 + m_2 \sinh \chi_2, \end{aligned} \quad (22)$$

the equations of relativistic energy and momentum conservation. These are the equations with intimidating square roots if one writes them using explicit gamma factors, that is, with  $\cosh \phi_1 = (1 - u_1^2)^{-\frac{1}{2}}$ ,  $\sinh \phi_1 = (1 - u_1^2)^{-\frac{1}{2}} u_1$ , etc.

It is convenient, as in the Newtonian case, to first assume that the second particle is initially at rest. So  $\phi_2 = 0$ , and  $\cosh \phi_2 = 1$ ,  $\sinh \phi_2 = 0$ . To find  $\cosh \chi_2$  and  $\sinh \chi_2$ , we eliminate  $\chi_1$  by taking the terms involving  $\chi_2$  to the left hand side, then square the component equations, and take their difference. Using the identity  $\cosh^2 \theta - \sinh^2 \theta = 1$ , there remains the single equation

$$(m_1 \cosh \phi_1 + m_2 - m_2 \cosh \chi_2)^2 - (m_1 \sinh \phi_1 - m_2 \sinh \chi_2)^2 = m_1^2. \quad (23)$$

After expanding out, and using the identity again, eq.(23) simplifies to

$$(m_1 \cosh \phi_1 + m_2)(\cosh \chi_2 - 1) = m_1 \sinh \phi_1 \sinh \chi_2, \quad (24)$$

an equation linear in  $\cosh \chi_2$  and  $\sinh \chi_2$ . Squaring this, and using the identity once more, gives

$$\begin{aligned} &((m_1 \cosh \phi_1 + m_2)^2 - m_1^2 \sinh^2 \phi_1) \cosh^2 \chi_2 \\ &- 2(m_1 \cosh \phi_1 + m_2)^2 \cosh \chi_2 \\ &+ ((m_1 \cosh \phi_1 + m_2)^2 + m_1^2 \sinh^2 \phi_1) = 0, \end{aligned} \quad (25)$$

a quadratic equation for  $\cosh \chi_2$  alone.

As in the Newtonian case, one solution is known. It is  $\cosh \chi_2 = 1$ , corresponding to the particles missing each other. The other solution is therefore rational in the coefficients and is

$$\cosh \chi_2 = \frac{(m_1 \cosh \phi_1 + m_2)^2 + m_1^2 \sinh^2 \phi_1}{(m_1 \cosh \phi_1 + m_2)^2 - m_1^2 \sinh^2 \phi_1}. \quad (26)$$

It follows from eq.(24) that

$$\sinh \chi_2 = \frac{2(m_1 \cosh \phi_1 + m_2)m_1 \sinh \phi_1}{(m_1 \cosh \phi_1 + m_2)^2 - m_1^2 \sinh^2 \phi_1}. \quad (27)$$

By cancelling a factor of  $m_1^2$ , these formulae simplify to

$$\cosh \chi_2 = \frac{(\cosh \phi_1 + \frac{m_2}{m_1})^2 + \sinh^2 \phi_1}{(\cosh \phi_1 + \frac{m_2}{m_1})^2 - \sinh^2 \phi_1}, \quad \sinh \chi_2 = \frac{2(\cosh \phi_1 + \frac{m_2}{m_1}) \sinh \phi_1}{(\cosh \phi_1 + \frac{m_2}{m_1})^2 - \sinh^2 \phi_1}, \quad (28)$$

showing that the outcome of a collision only depends on the mass ratio of the colliding particles. The outgoing velocity of particle 2 is

$$v_2 = \tanh \chi_2 = \frac{2(\cosh \phi_1 + \frac{m_2}{m_1}) \sinh \phi_1}{(\cosh \phi_1 + \frac{m_2}{m_1})^2 + \sinh^2 \phi_1}. \quad (29)$$

$\cosh \chi_1$  and  $\sinh \chi_1$ , and hence  $v_1$ , can be found using eqs.(22).

If the initial data are rational, then  $\cosh \chi_1$ ,  $\sinh \chi_1$ ,  $\cosh \chi_2$  and  $\sinh \chi_2$  are all rational. The outgoing velocities  $v_1$  and  $v_2$ , and their gamma factors, are therefore rational too. The expressions (28) for  $\cosh \chi_2$  and  $\sinh \chi_2$  have the form (14) associated with a Pythagorean triple, with

$$(M, N) = L \left( \cosh \phi_1 + \frac{m_2}{m_1}, \sinh \phi_1 \right). \quad (30)$$

Generally,  $\cosh \phi_1 + \frac{m_2}{m_1}$  and  $\sinh \phi_1$  are not integers, and one must multiply by a common factor  $L$  to find the integers  $(M, N)$  of a primitive triple.

As an example, suppose  $m_1 = 2$  and  $m_2 = 1$ , and the particles have initial velocities  $u_1 = \frac{3}{5}$  and  $u_2 = 0$ . The velocity  $u_1$  is associated to the Pythagorean triple  $(4, 3, 5)$ , with  $(\cosh \phi_1, \sinh \phi_1) = (\frac{5}{4}, \frac{3}{4})$ . From eq.(30),  $(M, N) = (7, 3)$ . So, from eqs.(28) and (29),  $(\cosh \chi_2, \sinh \chi_2) = (\frac{29}{20}, \frac{21}{20})$ , corresponding to the Pythagorean triple  $(20, 21, 29)$ , and  $v_2 = \frac{21}{29}$ . Further, from eq.(22),  $(\cosh \chi_1, \sinh \chi_1) = (\frac{41}{40}, \frac{9}{40})$ , corresponding to the triple  $(40, 9, 41)$ , and  $v_1 = \frac{9}{41}$ . (Another simple example is with  $u_1 = \frac{4}{5}$ ,  $u_2 = 0$  and masses  $m_1 = 3$  and  $m_2 = 1$ .)

It is not necessary to assume that particle 2 is initially at rest. The outgoing velocities and gamma factors are still rational for any rational initial data. This is because if initially  $u_1$  and  $u_2$  are nonzero, we can perform a Lorentz boost to make  $u_2$  vanish, calculate the result of the collision as above, and then apply the inverse boost. The boost, which acts on each 2-momentum in the same way, is the rational matrix

$$\begin{pmatrix} \cosh \phi_2 & -\sinh \phi_2 \\ -\sinh \phi_2 & \cosh \phi_2 \end{pmatrix}, \quad (31)$$

and reversing the sign of  $\phi_2$  gives the inverse boost. So the calculations preserve rationality at each step. More generally, the notion of rational



relativistic collision is invariant under any boost in the rational subgroup of the Lorentz group  $SO(1,1)$ .

It is also useful to consider the collision in the centre of mass frame. From general initial data one finds this frame by calculating the total 2-momentum

$$\begin{aligned} P = (E, p) &= (m_1\gamma(u_1) + m_2\gamma(u_2), m_1\gamma(u_1)u_1 + m_2\gamma(u_2)u_2) \\ &= (m_1 \cosh \phi_1 + m_2 \cosh \phi_2, m_1 \sinh \phi_1 + m_2 \sinh \phi_2) \end{aligned} \quad (32)$$

and then performing the Lorentz boost that makes  $p$ , the total spatial momentum, vanish. This boost is

$$(E^2 - p^2)^{-\frac{1}{2}} \begin{pmatrix} E & -p \\ -p & E \end{pmatrix}. \quad (33)$$

In the centre of mass frame, the particles just reverse their velocities in the collision, so  $v_1 = -u_1$  and  $v_2 = -u_2$ , and  $\gamma(v_1) = \gamma(u_1)$  and  $\gamma(v_2) = \gamma(u_2)$ . This solves eq.(21).

Therefore, in general, the outgoing particle 2-velocities are found by boosting the incoming 2-velocities to the centre of mass frame, reversing the signs of the spatial velocity components, and then inverting the boost. This combined operation is the improper Lorentz transformation

$$\begin{aligned} (E^2 - p^2)^{-1} \begin{pmatrix} E & p \\ p & E \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} E & -p \\ -p & E \end{pmatrix} \\ = (E^2 - p^2)^{-1} \begin{pmatrix} E^2 + p^2 & -2Ep \\ 2Ep & -(E^2 + p^2) \end{pmatrix}. \end{aligned} \quad (34)$$

For general, rational initial data,  $E$  and  $p$  are rational, and this matrix is too, even though the matrix (33) is not. The outgoing 2-velocities are therefore rational. The outgoing 2-velocity of particle 2 is

$$\begin{pmatrix} \cosh \chi_2 \\ \sinh \chi_2 \end{pmatrix} = (E^2 - p^2)^{-1} \begin{pmatrix} E^2 + p^2 & -2Ep \\ 2Ep & -(E^2 + p^2) \end{pmatrix} \begin{pmatrix} \cosh \phi_2 \\ \sinh \phi_2 \end{pmatrix}, \quad (35)$$

and similarly for particle 1. The outgoing velocity of particle 2 is

$$v_2 = \tanh \chi_2 = \frac{2Ep - (E^2 + p^2)u_2}{(E^2 + p^2) - 2Ep u_2}, \quad (36)$$

which agrees with eq.(29) when  $u_2 = 0$ .  $v_1$  is given by the same formula, with  $u_2$  replaced by  $u_1$ . Note that this does not mean that  $v_2$  is independent of  $u_1$ , nor  $v_1$  of  $u_2$ , because  $E$  and  $p$  depend on both  $u_1$  and  $u_2$ .

## 5 Collision as a 2-velocity reflection

In the centre of mass frame, the elastic collision of two particles reverses the signs of both spatial velocities. This is a spatial reflection in 2-velocity space. In a general frame of reference, the effect of the collision is the same reflection, conjugated by the Lorentz boost taking this frame to the centre of mass frame.

The conjugated reflection is the matrix (34), which acts linearly on 2-velocities as in eq.(35). Note that if a particle 2-velocity is  $U = (U^0, U^1) = (\gamma(u), \gamma(u)u)$  then  $u$  is the inhomogeneous, or projective, variable  $U^1/U^0$ . That is why the conjugated reflection acts on the particle velocity by the fractional linear transformation associated to the matrix (34), as we see in eq.(36).

To clarify its geometrical properties, let us suppress the particle label, and write the fractional linear transformation (36) as

$$v = \frac{2Ep - (E^2 + p^2)u}{(E^2 + p^2) - 2Ep u}. \quad (37)$$

This is an involution between  $u$  and  $v$ , a transformation whose square is the identity. Its orbits are pairs of points that are exchanged by the involution, together with two fixed points.

Generally, a fractional linear transformation

$$v = \frac{a - bu}{c - du} \quad (38)$$

can be rewritten as  $a - bu - cv + duv = 0$ . It is an involution if it is symmetric in  $u$  and  $v$ , in other words, if  $b = c$ , because the transformation from  $u$  to  $v$  is then the same as the transformation from  $v$  to  $u$ . Suppose the involution, which is now  $a - b(u + v) + duv = 0$ , has the further property of exchanging  $u = 1$  and  $v = -1$ . Then  $u + v = 0$ ,  $uv = -1$  is a solution, so  $d = a$ . The involution simplifies to

$$a - b(u + v) + auv = 0. \quad (39)$$

For the transformation (37), the coefficients are such that it simplifies in this way, with  $a = 2Ep$  and  $b = E^2 + p^2$ . The fixed points are found by setting  $v = u$ . They are  $u = \frac{p}{E}$  and  $u = \frac{E}{p}$ .

The transformation (37) is in fact completely determined by the conditions that (i) it is an involution, (ii) it maps  $-1$  and  $1$  into each other, and (iii) it has the centre of mass velocity  $\frac{p}{E}$  as a fixed point.

Physically, only velocities between  $-1$  and  $1$  occur. The involution transforms any velocity  $u$  in this range into a velocity  $v$  in the same range. The physically realisable fixed point is  $u = \frac{p}{E}$ , with  $|u| < 1$ . It describes the situation where one incoming particle is moving at the velocity of the centre of mass. The other incoming particle then has the same velocity, and the collision is at negligible relative speed. The outgoing particles have essentially the same velocities as the incoming ones.

Geometrically, an involution, if non-trivial, is always a type of reflection, but here we can be explicit, and show that the transformation (34) is a genuine reflection in 2-velocity space.

In the Euclidean plane, a reflection  $R$  is defined by

$$R(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n}, \quad (40)$$

where  $\mathbf{n}$  is a unit vector, satisfying  $\mathbf{n} \cdot \mathbf{n} = 1$ .  $R(\mathbf{x})$  is the reflection of  $\mathbf{x}$  in the line passing through the origin whose unit normal is  $\mathbf{n}$ .  $R(\mathbf{x})$  has the same length as  $\mathbf{x}$ , and  $R(R(\mathbf{x})) = \mathbf{x}$ .

In the 2-velocity plane with Lorentzian metric, let  $n$  be a spatial unit vector, satisfying  $n \cdot n = -1$ . The reflection defined by  $n$ , acting on a 2-velocity  $U$ , is

$$R(U) = U + 2(U \cdot n)n. \quad (41)$$

It is easy to verify that if  $U \cdot U = 1$  then  $R(U) \cdot R(U) = 1$ . Also  $R(R(U)) = U$ . In component form, the reflection is

$$\begin{pmatrix} R(U)^0 \\ R(U)^1 \end{pmatrix} = \begin{pmatrix} 1 + 2(n^0)^2 & -2n^0n^1 \\ 2n^0n^1 & 1 - 2(n^1)^2 \end{pmatrix} \begin{pmatrix} U^0 \\ U^1 \end{pmatrix}. \quad (42)$$

We require the vector  $n$  to be normal to  $P = (E, p)$ , the total 2-momentum, in the Lorentzian sense that  $n \cdot P = n^0E - n^1p = 0$ . This fixes  $n$  to be

$$n = (E^2 - p^2)^{-\frac{1}{2}}(p, E), \quad (43)$$

a unit vector that makes the same angle with the spatial momentum axis as  $P$  makes with the energy axis. For this  $n$  the matrix in (42) reproduces the matrix (34). It follows that if  $U$  is the incoming 2-velocity of either of the particles that collide, then its reflection  $V = R(U)$  is the outgoing 2-velocity of the particle.

A simple way to describe the effect of the collision is therefore to decompose the incoming 2-momenta as

$$m_1U_1 = \alpha P + \beta n, \quad m_2U_2 = (1 - \alpha)P - \beta n, \quad (44)$$

using  $P$  and  $n$  as basis 2-vectors. The outgoing 2-momenta are then

$$m_1 V_1 = \alpha P - \beta n, \quad m_2 V_2 = (1 - \alpha) P + \beta n. \quad (45)$$

Total 2-momentum is conserved, and each particle's 2-momentum remains on its mass-shell, because  $n \cdot P = 0$ .

This finally leads to a geometrical picture of the effect of the collision in the 2-momentum plane. The incoming 2-momenta,  $m_1 U_1$  and  $m_2 U_2$ , are on the mass-shell hyperbolae for particles of masses  $m_1$  and  $m_2$ . The total 2-momentum  $P$  is the vector sum of these. The outgoing 2-momenta,  $m_1 V_1$  and  $m_2 V_2$ , are on the same mass-shells and are the reflections of the incoming 2-momenta in the line through the origin,  $O$ , and  $P$ . These reflections are in the direction  $n$ , the Lorentzian normal to  $OP$ . The 2-momenta are shifted to the opposite ends of diameters of the hyperbolae parallel to  $n$ . These shifts are of equal magnitude but opposite sense, because  $P$  is the total momentum, and because the line  $OP$  bisects these diameters [3]. A typical collision is illustrated in Figure 1.

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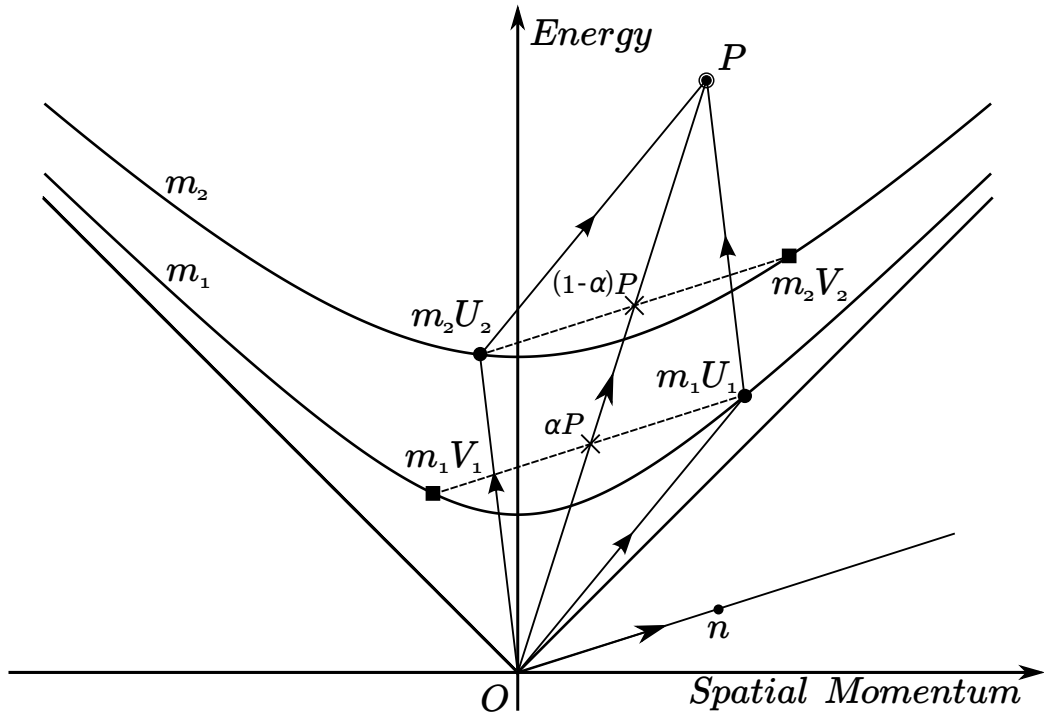


Figure 1: 2-momentum diagram for a relativistic, two-particle elastic collision. The outgoing 2-momenta  $m_1V_1$  and  $m_2V_2$  are obtained by a Lorentzian reflection of the incoming 2-momenta  $m_1U_1$  and  $m_2U_2$  in the line through  $O$  and the total 2-momentum  $P$ . The hyperbolae are the  $m_1$  and  $m_2$  mass-shells.  $n$  is the Lorentzian unit normal to  $OP$ .  $\alpha$  is the fraction of  $P$  carried by the first particle.